On a theorem concerning to univalent functions linearly accessible by M. Bienacki

(English translation of "*Sur un théorème concernant les fonctions univalentes linéairement accessibles de M. Bienacki*" by A. Bielecki and Z. Lewandowski)

1. A complex function

$$f(z) = a_1 z + a_2 z^2 + \cdots, \qquad a_1 \neq 0, \tag{1}$$

holomorphic in the unit disk $C_1 = \{z : |z| < 1\}$, is called *close-to-convex* if there exists a function $g(z) = b_1 z + b_2 z^2 + \cdots$, $b_1 \neq 0$, univalent and convex in C_1 such that

$$\operatorname{Re} \frac{f'(z)}{g'(z)} > 0, \qquad |z| < 1. \tag{2}$$

This conditions are fulfilled, then the function f must be univalent in $\mathbb{D}$ and

$$\operatorname{Re} \left\{ 1 + \frac{zg''(z)}{g'(z)} \right\} > 0, \qquad |z| < 1. \tag{3}$$

Close-to-convex functions were introduced in the theory of univalent functions by Kaplan in 1952 [3, p.169]. Z. Lewandowski noted that the notion of close-to-convex functions was defined already in 1936 by M. Bienacki [2] as a linearly accessible function. A function f of the form (1) is called linearly accessible if the complementary set of $f(C_1)$ in the plane can be covered by the closed half-lines do not intersect in pairs, which means that a point which belongs to two distinct half-lines must be the end point of at least one of them.

However, Lewandowski has shown that any linearly accessible functions are close-to-convex [4], and conversely, every close-to-convex functions are linearly accessible [5]. The proof of this last theorem, given in [5], is quite long and painful. The purpose of this note is to give another demonstration shorter, based on a simple principle that we used in [1].

2. Suppose that f is a function of the form (1) close-to-convex in $\mathbb{D}$ and set

$$F(z,t) = f(z) + tzg'(z)$$

for $z \in \mathbb{D}$ and $t \in [0, \infty)$. Since

$$\frac{\partial_z F(z,t)}{g'(z)} = \frac{z\partial_z F(z,t)}{\partial_t F(z,t)} = \frac{f'(z)}{g'(z)} + t\left[1 + \frac{zg''(z)}{g'(z)}\right],$$

it follows from (2) and (3) that the function $F(z,t)$ is close-to-convex univalent in $\mathbb{D}$ for any fixed $t \in [0, \infty)$, and in addition, we have $\operatorname{Re} z\partial_z F(z,t)/\partial_t F(z,t) > 0$ for all $z \in \mathbb{D}$ and $t \in [0, \infty)$[1], the following property of the function F is obtained (cf. [1, p.47]):

[1]This condition is interpreted as follows: When the paprameter t increases, the boundary curve of $f(C_\rho, t), \rho \in (0,1)$, moves so that the direction of the instantaneous velocity of any points P of the curve forms an acute angle with the conducted outside the normal curve (vector) at point P. Thus widening the domain $F(C_\rho, t)$ as t increases.

Property 1. *If $\rho \in (0,1)$ and $t_1 \le t_2$, and if $C_\rho = \{z : |z| < \rho\}$, then the domain $F(\overline{C_\rho}, t_1) = \{\zeta : \zeta = F(z,t_1), |z| \le \rho\}$ is contained in the domain $F(C_\rho, t_2)$.*

3. Fix an integer $n \ge 2$ and assume that

$$r = 1 - \frac{1}{n}, \qquad f_n(z) = f(rz). \tag{4}$$

It is seen that for $t \ge 0$

$$\Gamma(t) = \{\zeta : \zeta = F(re^{i\theta}, t), \theta \in [0, 2\pi)\}$$

is a simple curve which is the boundary of the domain $F(C_r, t)$, while all

$$\ell(\theta) = \{\zeta : \zeta = F(re^{i\theta}, t),\ t \ge 0\},\ \ \theta : \text{fixed}$$

is a closed half-line whose endpoint is $f(re^{i\theta}) = f_n(e^{i\theta})$ is located on the boundary curve $\Gamma(0)$ of the domain $f_n(C_1)$.

Suppose $0 \le |\theta - \sigma| < 2\pi$ and the half-lines $\ell(\theta)$ and $\ell(\sigma)$ have a common point

$$\zeta = F(re^{i\theta}, t) = F(re^{i\sigma}, s).$$

The function $F(z,t)$ is univalent for any fixed $t \ge 0$ and $t \ne s$. But this leads to a contradiction, because in this case one curve $\Gamma(t)$ or $\Gamma(s)$ must be contained in the interior of the other, under Property 1. We have shown that for $\theta \in [0, 2\pi)$, the half-lines $\ell(\theta)$ are disjoint in pairs. Note also none of the rays $\ell(\theta)$ may pass through the points contained in the interior of the curve $\Gamma(0)$, if a ray $\ell(\theta)$ should cut the curve at a point $f(re^{i\tau})$, where $\tau \ne \theta$, which would be the endpoint of another half-line $\ell(\tau)$.

4. Now fix a point ζ located outside the circle $\Gamma(0)$, and denote by $\psi(\theta)$ the angle between the real axis and the ray $m(\theta)$ from the point $f(re^{i\theta})$ and passing through the point ζ, and $\varphi(\theta)$ is the angle between the real axis and the ray $\ell(\theta)$. However, under the definition of $\ell(\theta)$, $\varphi(\theta) = \arg \partial_t F(re^{i\theta}) = \arg\{re^{i\theta} g'(re^{i\theta})\}$.

But the function g is convex and therefore increases with the angle $\varphi(\theta)$ and θ and $\varphi(\theta + 2\pi) = \varphi(\theta) + 2\pi$. On the other hand, because of $\psi(\theta + 2\pi) = \psi(\theta)$, ζ is located on the contour (boundary?) $\Gamma(0)$. So the increase of the angle $\chi = \varphi(\theta) - \psi(\theta)$ corresponding to the increase 2π of the parameter θ is equal to 2π, and therefore, there is a real number θ_0 and an integer k such as $\chi(\theta_0) = 2\pi k$; but this means that the ray $\ell(\theta_0) = m(\theta_0)$ passes through the point ζ.

We have demonstrated the lines $\ell(\theta)$, where $\theta \in [0, 2\pi)$ cover all the points outside the contour $\Gamma(0)$ limiting the domain $f_n(C_1) = f(C_r)$. Points belonging to the same time range are the origins of rays $\ell(\theta)$ and in Section 3 we found that these rays do not intersect each other. So the function $f_n(z)$ is linearly accessible from the definition due to Bienacki, and this result is obviously true for $n = 2, 3, \cdots$.

According to (4), the sequence of functions $\{f_n\}$ converges uniformly to the function f in any circle C_ρ, where $\rho \in (0,1)$, and these functions are linearly accessible. Under a theorem of Bienacki that is enough for the limit function f is also linearly accessible, which completes our proof.

References

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